Optimization of Cascade Blade Mistuning, Part II: Global Optimum and Numerical Optimization

E. Nissim* and R. T. Haftka†
Virginia Polytechnic Institute and State University, Blacksburg, Virginia

The values of the mistuning which yield the most stable eigenvectors are analytically determined, using the simplified equations of motion which were developed in Part I of this work. It is also shown that random mistunings, if large enough, may lead to the maximal stability, whereas the alternate mistunings cannot. The problem of obtaining maximum stability for minimal mistuning is formulated, based on numerical optimization techniques. Several local minima are obtained using different starting mistuning vectors. The starting vectors which lead to the global minimum are identified. It is analytically shown that all minima appear in multiplicities which are equal to the number of compressor blades. The effect of mistuning on the flutter speed is studied using both an optimum mistuning vector and an alternate mistuning vector. Effects of mistunings in elastic axis locations are shown to have a negligible effect on the eigenvalues. Finally, it is shown that any general two-dimensional bending-torsion system can be reduced to an equivalent uncoupled torsional system.

Introduction

THE aerodynamic energy method is further discussed and a set of approximate equations is developed in this work which permit the analytical determination of mistuning vectors which lead to maximum stability. It involves mistunings having peak values of 6%. A problem is then posed with the objective of obtaining maximum stability with smaller values of mistuning. This is done by combining the physical insight gained in both parts of this work,[1] with a numerical optimization algorithm. All the numerical results presented in this work are obtained by solving the "exact" equation of motion. The simplified equations of motion are used for analytical work only. The analytical results are then tested using the exact equations of motion.

Determination of the Ideal Eigenvectors for an Optimal System

On the basis of the discussion presented in Part I of this work,[1] it can be argued that the interference effects that lead to flutter can further be reduced if mistuning can be introduced, such that the resulting angle-of-attack $\{\alpha\}$ eigenvectors yield a single oscillating blade, with other blades stationary (or one order of magnitude smaller relative to the oscillating blade). This is true since, in this case, not only the interference effects of the adjoining blades is avoided, but also the interference phase effects of all the other blades. If such mistuning is physically possible, the corresponding generalized $\{q\}$ vectors can be seen from Eq. (1) [see Eq. (73) of Ref. 1] to coincide with the transformation matrix. That is, for $\alpha_1 = 1$ and all the other α's zero, the corresponding $\{q\}$ vector is identical with the first column of the transformation matrix in Eq. (1)

$$\{q\} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i\beta_1} & e^{i\beta_2} & & e^{i\beta_{N-1}} \\ 1 & e^{i2\beta_1} & e^{i2\beta_2} & & e^{i2\beta_{N-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{i(N-1)\beta_1} & e^{i(N-1)\beta_2} & & e^{i(N-1)\beta_{N-1}} \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{Bmatrix} \tag{1}$$

where β_r is the phase angle between neighboring blades defined by

$$\beta_r = (2\pi/N)r; \quad r = 0, \quad 1, \ldots, \quad N-1 \tag{2}$$

and N is the number of blades. Similarly, for $\alpha_j = 1$ with all the other α's zero, the corresponding $\{q\}$ vector is identical with the jth column of the transformation matrix in Eq. (1). This indicates that the response vectors are such that the moduli of all the q's are constant, i.e.,

$$|q_i| = |q_j| = \text{const} \quad j = 1,2,\ldots N; \quad i = 1,2,\ldots,N \tag{3}$$

Equation (3) holds for all eigenvectors, and the constant is the same for all eigenvectors. This result is very interesting since reference to the energy equation [Eq. (69)[1]] indicates that, in this case, the dissipation of energy per one cycle of oscillation is the same for all the eigenvectors. Therefore, following the discussion of the aerodynamic energy presented earlier in this work[1] means that no further improvement can be obtained through coupling, thus indicating that this mistuning achieves the maximum stability. In future sections of this work, these eigenvectors will be referred to as the optimal eigenvectors. It should, however, be stated that in the preceding discussion some relaxation has been made whereby small motions for the nondominant blades are permitted, with the hope of making these responses physically realizable.

Analytical Determination of Mistunings Associated with Optimal Eigenvectors

Having obtained analytically the hypothetical eigenvectors that, if found physically realizable, relate to the optimally

Received Sept. 13, 1983; revision received May 21, 1984.

*Visiting Professor (on leave from Technion, Israel Institute of Technology). Member AIAA.
†Professor. Member AIAA.

mistuned system, attempts can now be made to use these known vectors in order to obtain (analytically) both the mistunings associated with the optimal eigenvectors and the variation of these eigenvectors associated with mistuning. Consider (with no loss of generality) stiffness mistunings with zero structural damping (i.e., $g=0$), so that the simplified equation of motion developed in Part I[1] (Eq. (45) becomes relevant to our analysis, i.e.,

$$\bar{\lambda}\{\alpha\} = \frac{1}{2}(1+ig)\left[\frac{\Delta\omega_\alpha^2 I_\alpha}{100}\right] + \frac{1}{2M_R\bar{r}_\alpha^2}[\,[E]\,(4\bar{U}[C_1]$$

$$-i4\bar{U}^2[C_0] + i4[C_2]\,)\,[E]^{-1}\,]\{\alpha\} \qquad (4)$$

where M_R is the mass ration, $\bar{r}_\alpha$ the nondimensional radius of gyration of a single blade, $[\Delta\omega_\alpha^2 I_\alpha]$ a diagonal matrix representing percentage of stiffness mistunings, $\bar{U}$ the nondimensional airspeed $[E]^{-1}$ is given by the transformation matrix in Eq. (1) (except for a factor N), $\bar{\lambda}$ is a reduced nondimensional eigenvalue given by

$$\bar{\lambda} = \mu + i\nu - i \qquad (5)$$

and $[C_0]$, $[C_1]$, $[C_2]$ are complex aerodynamic matrices.

At this stage, some further discussion regarding the nature of the aerodynamic matrices is necessary. It has already been stated (in Part I) that the aerodynamic matrices are of the second order compared with the stiffness and mass matrices. Furthermore, these second-order aerodynamic terms lie along the diagonal, whereas the two nearest off-diagonal terms (one on each side of the diagonal) are smaller by one additional order of magnitude. All the other terms in the aerodynamic matrices are smaller by at least two additional orders of magnitude, when compared with the diagonal terms. Furthermore, due to the cyclic symmetry, all these aerodynamic diagonal terms are identical. Hence, values such as 2% stiffness mistuning bring the diagonal terms in Eq. (4) to about the same values as the diagonal aerodynamic terms. Ignoring smaller order terms in the matrices on the right-hand side of Eq. (4), while attempting to determine the eigenvectors, one ends up with a diagonal lambda matrix. This diagonal matrix cannot be used to determine the real parts of $\bar{\lambda}$, since these real parts have values of the third order (and comparable values were ignored in the lambda matrix). However, the simplified diagonal lambda matrix, as derived from Eq. (4), can be used to compute the imaginary parts of $\bar{\lambda}$, which now represent an approximation to the eigenvalues, and the eigenvectors. Based on the approximation just described, Eq. (4) can be written as

$$\bar{\lambda}\{\alpha\} = \frac{i}{2}\left(\left[\frac{\Delta(\omega_\alpha^2 I_\alpha)}{100}\right] + a(\epsilon^2)\,\lceil I \rfloor\right)\{\alpha\} \qquad (6)$$

where $a(\epsilon^2)$ is a constant (which depends on $\bar{U}$), of order ϵ^2, which represents the imaginary part of the diagonal elements involving the aerodynamic matrices. As already stated earlier, these aerodynamic diagonal terms are all equal due to the nature of the cyclic symmetry of compressor blade problems.

Assume now that $\bar{\lambda}_s$ is determined using the sth equation (in the set of equations) represented by Eq. (6), and assume that its value is substituted back in Eq. (6) in order to determine the α eigenvector associated with this value of $\bar{\lambda}_s$. The jth row of Eq. (6) yields, therefore,

$$\left(\bar{\lambda}_s - \frac{i}{2}\left[\frac{\Delta(\omega_\alpha^2 I_\alpha)_j}{100} + a(\epsilon^2)\right]\right)\alpha_j = 0,\; j=1,2,\ldots N \qquad (7)$$

Hence, if $\bar{\lambda}_s$ is such that the coefficient of α_j vanishes, then $\alpha_j \neq 0$. If, however, the coefficient of α_j is other than zero, then $\alpha_j = 0$. At this stage, a qualification must be introduced to the last statement made. Since Eq. (7) is obtained by ignoring terms of order ϵ^3, any nonvanishing coefficient of α_j will yield $\alpha_j = 0$, provided this latter coefficient is not of the order of the approximation made, and, therefore, these nonvanishing coefficients must be of order ϵ^2. These conditions are summarized by the following equations.

$$\alpha_j \neq 0 \text{ if } \bar{\lambda}_s - \frac{i}{2}\left[\frac{\Delta(\omega_\alpha^2 I_\alpha)_j}{100} + a(\epsilon^2)\right] = 0 \qquad (8)$$

$$\alpha_j = 0 \text{ if } \left|\bar{\lambda}_s - \frac{i}{2}\left[\frac{\Delta(\omega_\alpha^2 I_\alpha)_j}{100} + a(\epsilon^2)\right]\right| \geq \epsilon^2 \qquad (9)$$

and

$$\Delta(\omega_\alpha^2 I_\alpha)_j/100 \geq \mathcal{O}(\epsilon^2)$$

Equations (8) and (9) can be simplified if the solution for $\bar{\lambda}_s$ is substituted in them. However, since

$$\bar{\lambda}_s = \frac{i}{2}\left[\frac{\Delta(\omega_\alpha^2 I_\alpha)_s}{100} + a(\epsilon^2)\right] \qquad (10)$$

one can write

$$\alpha_j \neq 0 \text{ if } \Delta(\omega_\alpha^2 I_\alpha)_s = \Delta(\omega_\alpha^2 I_\alpha)_j \qquad (11)$$

$$\alpha_j = 0 \text{ if } \frac{1}{2}\left|\frac{\Delta(\omega_\alpha^2 I_\alpha)_s}{100} - \frac{\Delta(\omega_\alpha^2 I_\alpha)_j}{100}\right| \geq 0.01$$

or, alternatively,

$$\alpha_j = 0 \text{ if } |\Delta(\omega_\alpha^2 I_\alpha)_s - \Delta(\omega_\alpha^2 I_\alpha)_j| \geq 2 \qquad (12)$$

As stated earlier, $\Delta(\omega_\alpha^2 I_\alpha)_s$ represents the percentage mistuning in stiffness associated with the jth blade.

Let us now consider the case of purely alternate mistuning and assume, only for purposes of illustration, that blade 1 is associated with positive mistuning. Hence, for any odd value of s, Eq. (11) is satisfied by all odd blades (i.e., odd j values) and Eq. (12) is satisfied by all even blades (i.e., even j values). If an even value of s is chosen, then Eq. (11) is satisfied by all even blades and Eq. (12) is satisfied by all odd blades. These results indicate that in all cases, only alternate blades will participate in the motion whereas intermediate blades will be stationary. However, since this purely alternate mistuning preserves a cyclic symmetry (N even), it follows that the amplitudes of all moving blades must be equal, with phase differences of 2 β_r between adjoining moving blades. This result was found earlier.[1] However, the importance of Eqs. (11) and (12), as they relate to this purely alternate mistuning, is that they show that this result is maintained irrespective of any increase in the purely alternate mistuning. The only effect of any increase in this type of mistuning is associated with the improvement in the approximation made in the present analysis, and therefore, the motion of these "approximately stationary" blades will become even smaller as the purely alternate mistuning is increased. Hence, one arrives at the important conclusion that the purely alternate mistuning can never lead to the maximum stability irrespective of the amount of mistuning introduced.

Let us now consider Eqs. (11) and (12) in relation to the optimal eigenvector. In this case we wish to define the values of the different $\Delta(\omega_\alpha^2 I_\alpha)_j$ terms so that for any chosen value of s, only $\alpha_s \neq 0$ [i.e., $j=s$ satisfies Eq. (11)] whereas all the other α_j are zero. Since Eq. (11) is always satisfied for $j=S$ [provided $\Delta(\omega_\alpha^2 I_\alpha)_s/100 \geq \mathcal{O}(\epsilon^2)$], all one needs now is to satisfy Eq. (12). For large amounts of mistunings, it should not be difficult to satisfy Eq. (12). Hence, there exists an infinite number of mistuning values which, when made large enough, lead to the maximum stability. However, for relatively small amounts of mistunings, one can still satisfy Eq. (12), provided some thought is given, during the process of selection of the mistuned values, to the following points.

1) For any single moving blade, the strongest aerodynamic interference is caused by its two immediate neighbors (one on each side). Hence, it is of major importance to decrease these interference effects by reducing the motion of these two neighboring blades. As a result, one should aim at reducing primarily the motion of these two neighboring blades; and only then should one attempt the reduction of the motion of all other blades be made.

2) As a consequence of point 1, one should use mistunings with alternate signs, since, in this latter case, the left-hand side of Eq. (12) will yield the largest possible values (with given amounts of mistunings) both for blade $(s+1)$ and for blade $(s-1)$.

3) All values of mistunings which are of the same sign must have different numerical values so as to satisfy Eq.(12). However, since these equal-sign values of mistuning relate (as a result of the alternate sign mistuning) to blades which are not in the immediate neighborhood of the moving blade, some relaxation can be introduced in Eq. (12). Thus, one might use the value of 1 (instead of 2) in Eq. (12), leading to the satisfaction of Eq. (12) for blades associated with mistuning having signs opposite to that of $\Delta(\omega_\alpha^2 I_\alpha)_s$ (which includes the two blades, one on each side of the moving blade s) and to the following relaxation

$$|\Delta(\omega_\alpha^2 I_\alpha)_s - \Delta(\omega_\alpha^2 I_\alpha)_j| \geq 1, \quad j = s\pm 2,\ s\pm 4,\ldots \tag{13}$$

which affects only blades which are not immediately adjacent to the moving s blade.

The simplest example of mistunings which relate to values of $\Delta(\omega_\alpha^2 I_\alpha)$, and that satisfy Eqs. (11-13) (for our 12-bladed example treated earlier) is given by -1, 1, -2, 2, -3, 3, -4, 4, -5, 5, -6, 6. These values of mistuning will be referred to as the first-synthetic set. The eigenvalues obtained using this set of mistunings are given in Table 1. (Using the same example employed in Refs. 1 and 3; for the sake of completeness, it will be described again in a subsequent section of this work). Comparison between the results presented in Table 4 of Part I for the purely alternate mistunings (with $\pm 6\%$ values), and those presented in Table 1 for the optimal mistuning (also with peak mistuning of 6%), indicates the excellence of the results associated with the first synthetic set of mistunings. One can mix the first-synthetic vector in any random order, while maintaining the alternate signs, and obtain a similar excellent quality of results [note that such a "mixing" of terms maintains the satisfaction of Eqs. (11-13)]. It should also be noted that the aforementioned first-synthetic set of mistunings maintains the constraint imposed regarding zero mean mistuning [see Eqs. (46) and (47) in Part I]. An example of such a mixed set, derived from the first-synthetic set, and denoted as the second-synthetic set, is given by the following values of $\Delta(\omega_\alpha^2 I_\alpha)$: -5, 2, -3, 3, -4, 5, -2, 4, -1, 6, -6, 1. In this example an attempt was made to arrange the alternate terms such that the right-hand side of Eq. (13) (Part I) is as large as possible for all blades. The results corresponding to the mistunings just given are presented in Table 2, and they, again, show excellent quality with even some slight improvement compared to Table 1. Hence, one can conclude that an infinite number of mistuning vectors exist which have maximum stability and that an increase of any such synthetic or optimal vector by a scalar multiplier will lead to some small improvements in the values of μ's, as a result of improving the approximations leading to Eqs. (11-13). Table 3 shows the results obtained by multiplying the first-synthetic vector by 2. These results support the analytical conclusions reached earlier regarding the increase of mistuning by a scalar

Table 1 Eigenvalues relating to first synthetic set of stiffness mistuning at $\bar{U}=0.5$, $\bar{U}=0.787$, $N=12$

First synthetic set	Eigenvalue at $\bar{U}=0.5$		Eigenvalue at $\bar{U}=0.787$	
of stiffness mistuning, %	Re	Im	Re	Im
−1	−0.247010D−02	0.964816D+00	−0.519666D−02	0.958164D+00
1	−0.245589D−02	0.969972D+00	−0.516405D−02	0.963337D+00
−2	−0.229675D−02	0.102500D+01	−0.472887D−02	0.101837D+01
2	−0.229403D−02	0.102011D+01	−0.467979D−02	0.101345D+01
−3	−0.245999D−02	0.975116D+00	−0.518264D−02	0.968573D+00
3	−0.247091D−02	0.980237D+00	−0.465608D−02	0.100843D+01
−4	−0.249823D−02	0.985342D+00	−0.522453D−02	0.973815D+00
4	−0.259401D−02	0.990458D+00	−0.532336D−02	0.979115D+00
−5	−0.228612D−02	0.101519D+01	−0.584343D−02	0.984747D+00
5	−0.221135D−02	0.100028D+01	−0.459599D−02	0.100332D+01
−6	−0.221979D−02	0.100525D+01	−0.440125D−02	0.998010D+00
6	−0.226792D−02	0.101024D+01	−0.414384D−02	0.993159D+00

Table 2 Eigenvalue relating to second synthetic set of stiffness mistunings at $\bar{U}=0.5$, $\bar{U}=0.787$, $N=12$

Second synthetic set	Eigenvalue at $\bar{U}=0.5$		Eigenvalue at $\bar{U}=0.787$	
of stiffness mistunings, %	Re	Im	Re	Im
−5	−0.246959D−02	0.964815D+00	−0.519498D−02	0.958149D+00
2	−0.248584D−02	0.969992D+00	−0.526884D−02	0.963512D+00
−3	−0.228407D−02	0.102499D+01	−0.517867D−02	0.968570D+00
3	−0.228264D−02	0.102010D+01	−0.530468D−02	0.973939D+00
−4	−0.226284D−02	0.101519D+01	−0.468058D−02	0.101832D+01
5	−0.228172D−02	0.101025D+01	−0.469229D−02	0.101331D+01
−2	−0.228024D−02	0.100529D+01	−0.449314D−02	0.100842D+01
4	−0.229554D−02	0.100032D+01	−0.465942D−02	0.100336D+01
−1	−0.246214D−02	0.975120D+00	−0.459752D−02	0.998380D+00
6	−0.248816D−02	0.980247D+00	−0.462022D−02	0.993472D+00
−6	−0.246882D−02	0.990382D+00	−0.520173D−02	0.978924D+00
1	−0.246350D−02	0.985321D+00	−0.524841D−02	0.984124D+00

multiplier. One could experiment with some relaxations in Eqs. (12) and (13), however, this will depend on the examples treated and the general analytical justification may be lost. Furthermore, for higher values of $\bar{U}$ (or substantially smaller values of M_R), some adjustments may be necessary in the 1 and 2 values appearing in Eqs. (12) and (13). These values will clearly depend on the configurations used. However, a simple order of magnitude analysis should readily yield the required adjustments.

Finally, the aerodynamic terms in Eq. (4) decrease in magnitude with a decrease in $\bar{U}$. It, therefore, follows that if Eqs. (10-13) are satisfied at a given value of $\bar{U}$, they will also be satisfied at all speeds below the given value of $\bar{U}$. Hence, one concludes that if the mistunings of a system yield maximal stability at $\bar{U}_{\text{opt}}$, the system will also be most stable at all speeds below $\bar{U}_{\text{opt}}$.

Computation of μ_{mean} Using Energy Equations

It has already been shown[1] that the effect of mistuning is to decrease the large values of μ and to increase the small values of μ so that, at the limit, all the μ tend to μ_{mean}. It has further been shown that the value of μ_{mean} is independent of the mistuning of the blades and that, as a result, it can be evaluated using the tuned blade system. This is very important since the maximal stability of the mistuned system can be predicted from the results relating to the tuned system. It will be shown below that μ_{mean} can be determined directly from the data of the tuned system and that there is even no need to solve for the eigenvalues of the tuned system.

It has alreay been shown that the eigenvector associated with maximal stability represents the vibration of a single blade only with all other blades essentially stationary. Hence, the energy dissipation during one cycle of oscillation of this single-bladed, single degree of freedom system (in the α domain) can be evaluated by means of an equivalent viscous damping coefficient C_{eq} where

$$P=\pi\omega C_{\text{eq}}\alpha^2 \tag{14}$$

and where α can be assumed to be real, with no loss of generality. However, P can be written in terms of aerodynamic coefficients, as shown by Eq. (69) (Part I). Hence, equating Eqs. (35) and (69)[1] one obtains

$$\pi\omega C_{\text{eq}}\alpha^2=\pi^2\rho U^2C^2N(q_R^2+q_I^2)\sum_{j=1}^{N}(-C_{M_{\alpha_j}})_I \tag{15}$$

where $(q_R^2+q_I^2)$ represents the q responses which are the same for all the generalized coordinates, as already shown earlier. Equation (15) can be reduced to

$$C_{\text{eq}}=\frac{\pi\rho U^2C^2N}{\omega}\,\frac{(q_R^2+q_I^2)}{\alpha^2}\sum_{j=1}^{N}(-C_{M\alpha_j})_I$$

and after dividing both sides of the preceding equation by $2mr_\alpha^2\omega$, one obtains

$$\frac{C_{\text{eq}}}{2mr_\alpha^2\omega}=\frac{2\pi\rho b^2}{m}\cdot\frac{1}{\bar{r}_\alpha^2}\cdot\frac{U^2}{\omega^2C^2}N\frac{(q_R^2+q_I^2)}{\alpha^2}\sum_{j=1}^{N}(-C_{M\alpha_j})_I$$

which reduces to

$$\mu=\frac{2N}{M_R\bar{r}_\alpha^2k_c^2}\,\frac{(q_R^2+q_I^2)}{\alpha^2}\sum_{j=1}^{N}(C_{M\alpha_j})_I \tag{16}$$

since in a single degree of freedom system

$$-(C_{\text{eq}}/2mr_\alpha^2\omega)=\mu \tag{17}$$

Substituting now for α in Eq. (66),[1] i.e., $\alpha_s=1$ and $\alpha_j=0$ for all $j\neq s$, and remembering Eq. (58),[1] one obtains

$$\frac{q_R^2+q_I^2}{\alpha^2}=\frac{1}{N^2} \tag{18}$$

Substituting Eq. (18) into Eq. (16) one, finally, obtains

$$\mu=\frac{2}{M_R\bar{r}_\alpha^2k_c^2}\sum_{j=1}^{N}\frac{(C_{M\alpha_j})_I}{N} \tag{19}$$

However, since the aforementioned limiting value of μ is equal to μ_{mean}, Eq. (19) can be rewritten as

$$\mu_{\text{mean}}=\frac{2}{M_R\bar{r}_\alpha^2k_c^2}(C_{M\alpha I})_{\text{mean}} \tag{20}$$

where

$$(C_{M\alpha_I})_{\text{mean}}=\sum_{j=1}^{N}(C_{M\alpha_j})_I\Big/N \tag{21}$$

Equation (20) is extremely useful and it will now be tested using the values of $C_{M\alpha_I}$'s for $k_c=2.05$ in Table 2 of Part I. This value for k_c is slightly larger than the value of k_c at flutter (around $k_c\approx 2$) but it is close enough for comparison with results obtained with $\bar{U}=0.5$ and which appear in Table 3 (Part I). Substituting $M_R=258.5$, $\bar{r}_\alpha=0.2887$, $k_c=2.05$ and $(C_{M\alpha_I})_{\text{mean}}=-0.10334$ (from Table 1, Part I) into Eq. (20) one obtains $\mu_{\text{mean}}=-0.00228$. However, the value of μ_{mean}, as obtained from Table 3[1] (for the tuned system) is -0.00238. This is an excellent agreement when considering the differences in k_c mentioned earlier. Hence, one concludes that for any value of k_c chosen, one can evaluate the maximal stability of the mistuned system using only the data of the system in conjunction with Eq. (20).

Table 3 Eigenvalues relating to twice the first synthetic set of stiffness mistunings at $\bar{U}=0.5$, $\bar{U}=0.787$, $N=12$

First synthetic set of stiffness mistuning ×2, %	Eigenvalue at $\bar{U}=0.5$ Re	Eigenvalue at $\bar{U}=0.5$ Im	Eigenvalue at $\bar{U}=0.787$ Re	Eigenvalue at $\bar{U}=0.787$ Im
−2	−0.248194D−02	0.933207D+00	−0.466436D−02	0.104753D+01
2	−0.247152D−02	0.943855D+00	−0.469801D−02	0.103798D+01
−4	−0.246353D−02	0.954387D+00	−0.470994D−02	0.102827D+01
4	−0.246929D−02	0.985323D+00	−0.471180D−02	0.101844D+01
−6	−0.247334D−02	0.975125D+00	−0.468704D−02	0.100848D+01
6	−0.246108D−02	0.964807D+00	−0.452723D−02	0.998310D+00
−8	−0.227815D−02	0.105384D+01	−0.520788D−02	0.926132D+00
8	−0.229359D−02	0.104432D+01	−0.523919D−02	0.978993D+00
−10	−0.226217D−02	0.100528D+01	−0.522747D−02	0.968641D+00
10	−0.229668D−02	0.101520D+01	−0.518951D−02	0.936898D+00
−12	−0.229931D−02	0.103471D+01	−0.517365D−02	0.947556D+00
12	−0.230194D−02	0.102500D+01	−0.517561D−02	0.958121D+00

Formulation of the Constraints and Objective Function

In Ref. 1, it was shown that the mean value of the real part of the flutter eigenvalues, μ_{mean}, is not affected by mistuning

$$\mu_{mean} = \frac{1}{N} \sum_{i=1}^{N} \mu_i \tag{22}$$

where μ_i is the real part of the ith eigenvalue. It was further shown that maximum stability is achieved when all eigenvalues have the same real part. The amount of mistuning which is required depends on how close it is desired to approach the maximum stability limit.

The objective of the following sections of this work is to achieve high stability with small amounts of mistuning. There are several possible formulations of this problem as an optimization problem. It is possible, for example, to specify the magnitude of mistuning and seek the mistuning distribution that maximizes stability. Alternatively, it is possible to specify how close it is desired to get to the limiting value and seek the minimal amount of mistuning that satisfies the stability limit. This latter approach was selected here because of computational considerations discussed later. Therefore, the stability constraint is introduced by specifying the maximum distance of μ_{max} from μ_{mean}, that is

$$\mu_{max} - \mu_{mean} \leq r\,|\mu_{mean}| \tag{23}$$

where $r<1$. In addition to the constraint just given, the zero mean mistuning requirement[1] is introduced in the form of inequality constraints since the optimization algorithm NEWSUMT[2] employs only inequality constraints. To prevent ill conditioning, the zero mean mistuning condition is relaxed slightly, that is,

$$-0.05 \leq \sum_{s=1}^{N} \frac{\Delta I_{\alpha s}}{N} \leq 0.05 \tag{24}$$

$$-0.05 \leq \sum_{s=1}^{N} \frac{\Delta(\omega_\alpha^2 I_\alpha)_s}{N} \leq 0.05 \tag{25}$$

where $\Delta I_{\alpha s}$ denotes the percentage mass mistuning relating to the sth blade. Equations (24) and (25) imply that the deviations of the mean mistuning is permitted to vary by ± 0.05 of 1%, which is considered to be reasonably small.

It is desired to satisfy the constraints [Eqs. (24) and (25)] with a minimum amount of mistuning. The definition of minimum amount of mistuning is not clear cut. In this work this minimum amount, or the objective function, is given by

$$F_{min} = \left[\sum_{s=1}^{N} (\Delta I_{\alpha s})^2 + \sum_{s=1}^{N} (\Delta\omega_{\alpha s}^2 I_{\alpha s})^2 \right] \tag{26}$$

Another logical objective function is the maximum individual mistuning. Unfortunately, it is a function with discontinuous derivatives, and, therefore, difficult to minimize numerically. It can be approximated by a continuous function by replacing the squares in Eq. (26) by higher powers. The higher the power, the closer is the objective function to the maximum individual mistuning, however, its first derivatives then have very sharp changes. The use of higher powers in Eq. (26) is investigated later.

Optimization Routine

The NEWSUMT optimization routine is used in the numerical part of this work. It is described in Ref. 2 which also gives references to other works which form the theoretical basis of the NEWSUMT routine.

Description of the Numerical Example

All the results presented herein relate to the NASA test rotor 12 which is modified herein to include 12 blades instead of its 56 blades. This is the same example used in Part I of this work.[1] For the sake of completeness the main data are repeated herein. The elastic axis position and the c.g. position are assumed to be at the midchord point, unless otherwise stated. Additional data relates to the stagger angle $\xi = 54.4$ deg the radius of gyration (normalized with respect to the chord) $\bar{r}_\alpha = 0.2887$, the mass ratio $M_R = 258.5$, and the gap s between blades (normalized with respect to the chord) $s/c = 0.534$. The reduction in the number of blades is introduced in order to reduce the computational labor while testing the different aspects associated with mistuning. In addition, the blades are allowed rotational freedom only (instead of the bending torsion freedom allowed in Ref. 3). For elastic axis position at midchord, with c.g. also at midchord, the elimination of the bending degree of freedom is known to have negligible effects.[3] The case where the elastic axis does not coincide with the c.g. position will be treated later.

Difficulties Associated with the Mistuned System

The optimization routine is used to produce the optimum vector of the mistuned stiffness terms, i.e., the $\Delta(\omega_\alpha^2 I_\alpha)_s$ terms, at two different values of nondimensional speed $\bar{U}$, where $\bar{U}$ is given by

$$\bar{U} = U/\omega_\alpha c \tag{27}$$

where c denotes the chord of the blade. The airspeed values are chosen to be $\bar{U} = 0.5$ (which is at the flutter speed of the tuned system) and $\bar{U} = 0.787$ (which was originally used for purposes of comparison with Ref. 3). Before presenting the specific results obtained, two general remarks regarding the difficulties encountered during the optimization process are appropriate.

1) Many local minima exist and, therefore, the results obtained are highly dependent on the initial values of the mistuned vector.

2) The difficulties mentioned in 1 are somewhat alleviated since any minimum that exists appears in multiplicities which are equal to (at least) the number of blades N. This property is derived in Appendix A.

As a result of 1 and 2, the determination of the global minimum is a problem. For any set of minima obtained, the existence of other solutions which may yield even lower minimum values cannot be excluded. The search for a global minimum may be exhausting if treated on the basis of purely numerical experimentation. However, the analytical and numerical work presented above (and in Part I,[1]) plays a major role in the identification of the global minimum.

Optimization Results for the Mistuned System

Table 4 presents the numerical values for the optimum mistunings involving the stiffness terms (for $\bar{U} = 0.5$, $r = 0.1$), together with the eigenvalues associated with this optimum mistuning. For comparison, the eigenvalues associated with the tuned system are also included in Table 4. Note that the value of r used to obtain Table 4 (i.e., $r = 0.1$) imposes a severe constraint leading to a relatively highly stable system. It is important to mention that the results presented in Table 4 could be obtained only if the initial vector representing the mistunings had alternate signs (with values not necessarily equal). Initial values of mistunings, with nonalternating signs, lead to minima which were inferior to the one presented in Table 4. Table 4 indicates that small amounts of mistunings (all less than 2%) are sufficient to produce the required changes in the eigenvalues. This result conforms with predictions made earlier in this work,[1] which indicated that optimum mistunings could be achieved with changes introduced in the tuned values of the mass or stiffness terms, which are less than first-

Table 4 Optimum stiffness mistunings and optimized eigenvalues at $\bar{U}_{\text{opt}} = 0.5$, with eigenvalues for tuned system at same speed, $N = 12$

Optimum stiffness mistuning, %	Eigenvalue for optimum system		Eigenvalue for tuned system	
	Re	Im	Re	Im
−0.1154D+01	−0.254984D−02	0.987309D+00	−0.487309D−02	0.994687D+00
0.1295D+01	−0.262072D−02	0.988259D+00	−0.464336D−02	0.993961D+00
−0.6471D+00	−0.262022D−02	0.989696D+00	−0.387931D−02	0.993642D+00
0.1433D+01	−0.255669D−02	0.989863D+00	−0.443927D−02	0.995646D+00
−0.1115D+01	−0.267111D−02	0.992267D+00	−0.280948D−02	0.993772D+00
0.1585D+01	−0.263389D−02	0.992325D+00	−0.167639D−02	0.994288D+00
−0.6334D+00	−0.213857D−02	0.998841D+00	−0.709547D−03	0.995098D+00
0.1232D+01	−0.214341D−02	0.998914D+00	−0.103357D−03	0.995997D+00
−0.1437D+01	−0.216128D−02	0.100302D+01	0.250428D−04	0.996775D+00
0.7712D+00	−0.213859D−02	0.100242D+01	−0.384184D−03	0.997148D+00
−0.1621D+01	−0.214060D−02	0.100176D+01	−0.170643D−02	0.996767D+00
0.6802D+00	−0.213945D−02	0.100140D+01	−0.331665D−02	0.996550D+00

Table 5 Optimum stiffness mistuning and optimized eigenvalues at $\bar{U}_{\text{opt}} = 0.787$, together with eigenvalues for tuned system at same speed, $N = 12$

Optimum stiffness mistuning, %	Eigenvalue for optimum system		Eigenvalue for tuned system	
	Re	Im	Re	Im
−0.1387D+01	−0.537786D−02	0.982316D+00	−0.105878D−01	0.991488D+00
0.2858D+01	−0.517503D−02	0.968493D+00	−0.119345D−01	0.988625D+00
−0.2525D+01	−0.517442D−02	0.971978D+00	−0.114048D−01	0.986106D+00
0.2599D+01	−0.540472D−02	0.973027D+00	−0.938802D−02	0.984450D+00
−0.3239D+01	−0.528303D−02	0.976421D+00	−0.643798D−02	0.983936D+00
0.1366D+01	−0.523957D−02	0.976172D+00	−0.726562D−02	0.993815D+00
−0.3310D+01	−0.448912D−02	0.995182D+00	−0.320084D−02	0.984589D+00
0.3316D+01	−0.475553D−02	0.100934D+01	−0.406294D−03	0.986292D+00
−0.4012D+01	−0.445258D−02	0.100162D+01	0.140827D−02	0.988609D+00
0.4210D+01	−0.444915D−02	0.100251D+01	0.189490D−02	0.991002D+00
−0.2532D+01	−0.453002D−02	0.100516D+01	0.877237D−03	0.992744D+00
0.3241D+01	−0.479013D−02	0.100419D+01	−0.268256D−02	0.993199D+00

Table 6 Optimum results vs purely alternate results with same peak mistunings, $\bar{U}_{\text{opt}} = 0.5$, $N = 12$

Optimum stiffness mistuning, %	Eigenvalue for optimum system		Modulus of optimum mistuning	Eigenvalue for purely alternate system with ±1.621% mistuning		Modulus of ±1.62% mistuning
	Re	Im	$\lvert\alpha\rvert$	Re	Im	$\lvert\alpha\rvert$
−0.1165D+01	−0.254984D−02	0.987309D+00	0.1967D+00	−0.262627D−02	0.100326D+01	0.1747D−01
0.1295D+01	−0.262072D−02	0.988259D+00	0.2494D−01	−0.259889D−02	0.100356D+01	0.1126D+00
−0.6471D+00	−0.262022D−02	0.989696D+00	0.1345D−01	−0.227619D−02	0.100312D+01	0.1747D−01
0.1433D+01	−0.255669D−02	0.989863D+00	0.1362D−02	−0.174774D−02	0.100351D+01	0.1126D+00
−0.1115D+01	−0.267111D−02	0.992267D+00	0.1619D−01	−0.188513D−02	0.100327D+01	0.1747D−01
0.1585D+01	−0.263389D−02	0.992325D+00	0.1628D−02	−0.214913D−02	0.100349D+01	0.1126D+00
−0.6334D+00	−0.213857D−02	0.998841D+00	0.1340D−01	−0.236700D−02	0.986982D+00	0.1747D−01
0.1232D+01	−0.214341D−02	0.998914D+00	0.1030D−01	−0.270056D−02	0.987500D+00	0.1126D+00
−0.1437D+01	−0.216128D−02	0.100302D+01	0.2004D+00	−0.273344D−02	0.987403D+00	0.1747D−01
0.7712D+00	−0.213859D−02	0.100242D+01	0.6049D−01	−0.252288D−02	0.987416D+00	0.1126D−00
−0.1621D+01	−0.214060D−02	0.100176D+01	0.1603D+01	−0.239444D−02	0.987215D+00	0.1747D−01
0.6802D+00	−0.213945D−02	0.100140D+01	0.2816D+00	−0.251594D−02	0.987215D+00	0.1126D+00

Table 7 Optimum results vs purely alternate results with same peak mistunings, $\bar{U}_{\text{opt}} = 0.787$, $N = 12$

Optimum stiffness mistuning, %	Eigenvalue for optimum system		Eigenvalue for purely alternate system with ±4.21% mistuning	
	Re	Im	Re	Im
−0.1387D+01	−0.537786D−02	0.982316D+00	−0.559194D−02	0.101066D+01
0.2858D+01	−0.517503D−02	0.968493D+00	−0.604526D−02	0.100951D+01
−0.2525D+01	−0.517442D−02	0.971978D+00	−0.517843D−02	0.100863D+00
0.2599D+01	−0.540472D−02	0.973027D+00	−0.389121D−02	0.100875D+01
−0.3239D+01	−0.528308D−02	0.976421D+00	−0.319767D−02	0.100950D+01
0.1366D+01	−0.523957D−02	0.976172D+00	−0.410484D−02	0.101007D+01
−0.3310D+1	−0.448912D−02	0.995182D+00	−0.501942D−02	0.966608D+00
0.3316D+01	−0.475553D−02	0.100934D+01	−0.487964D−02	0.967290D+00
−0.4012D+01	−0.445258D−02	0.100162D+01	−0.531637D−02	0.967234D+00
0.4210D+01	−0.444915D−02	0.100251D+01	−0.495463D−02	0.967810D+00
−0.2532D+01	−0.453002D−02	0.100516D+01	−0.534204D−02	0.968141D+00
0.3241D+01	−0.479013D−02	0.100418D+01	−0.562297D−02	0.967900D+00

order changes. For first-order changes, 10% mistunings are required, and the 1.621% maximum mistuning in Table 4 is well within the predicted maximum.

Table 5 presents results similar to those shown in Table 4, but for $\bar{U}=0.787$. Here again, $r=0.1$ and an initially small vector of stiffness mistunings, with alternate signs, is used as a starting vector for the optimization. As expected, since the aerodynamic terms assume larger values due to the increase in $\bar{U}$, the amounts of mistunings increase accordingly However, the amounts of mistunings are still small, with values not exceeding 4.21%.

Comparison of Results with Optimum Mistunings and Purely Alternate Mistunings

Table 6 shows a comparison between the eigenvalues obtained using the optimum mistunings obtained with $\bar{U}=0.5$ and those obtained using purely alternate mistunings with values equal to the peak value of the optimum vector, i.e., with ±1.621% mistunings in stiffness. Table 7 shows a similar comparison but with $\bar{U}=0.787$. It can be seen that in both cases, the optimum vectors yield lower values for μ_{max} than the purely alternate mistuning vectors. These differences are about 16% and 26% of the respective values of μ_{mean}, and they indicate an inferior performance of the alternate mistuning vectors. Table 6 also presents typical moduli of the eigenvectors for the optimized system and for the purely alternate system with peak stiffness mistuning of 1.621%. Table 6 shows that the modulus of the optimized eigenvector yields essentially a single oscillating blade with all other blades effectively stationary (that is, at least one order of magnitude smaller). Hence, this latter eigenvector has the properties of the most stable eigenvector, as shown earlier in this work. Table 6 yields an eigenvector similar to the one obtained for the purely alternate system discussed in Ref. 1, i.e., each oscillating blade has two effectively stationary adjoining blades.

Flutter Speed of the Mistuned Systems

On the basis of the results obtained from the energy analysis,[1] the beneficial stabilizing effects introduced by the mistuning will be obtained for all speeds. Hence, it is impossible that mistuning will have destabilizing effects at a lower value of U than the one used to obtain the optimized mistuning (i.e., $\bar{U}_{opt}$). Table 8 summarizes the results obtained for the two aforementioned optimized mistunings, together with those obtained for the two purely alternate mistunings. It can be seen that in all cases, a dramatic increase in the flutter speed is obtained. The optimized system with $\bar{U}_{opt}=0.5$ shows a 66% increase in flutter speed ($\bar{U}_F=0.83$) and the system with purely alternate stiffness mistunings of ±1.621% shows an 86% increase in flutter speed ($\bar{U}_F=0.93$). For the optimized system with $\bar{U}_{opt}=0.787$ and the purely alternate system with peak values of ±4.21% the flutter speed could not be determined ($\bar{U}_F>1.0$) since they occurred at very low reduced frequencies which were far below the lowest reduced frequency (i.e., $k_c=1.15$) employed during the process of interpolation of the aerodynamic coefficients.

The reason for the superiority of the alternate system in yielding higher flutter speeds lies in the fact that optimization is performed at a given speed, say $\bar{U}_{opt}=0.5$, whereas performance is measured under widely different conditions pertaining to much higher air speeds. Hence the correct way of posing the problem should involve the choice of the maximum airspeed ($\bar{U}_{max}$) to which the system is going to be subjected. Optimization should then be performed at $\bar{U}_{max}$, with performance of the different mistuned systems compared at $\bar{U}_{max}$ or at lower speeds. Hence, the results presented in Table 6 should relate to $\bar{U}_{max}=0.5$, and those presented in Table 7 should relate to $\bar{U}_{max}=0.787$. Under these conditions, the optimized systems are superior, at $\bar{U}_{opt}$, compared with systems having purely alternate mistunings. Some values for μ_{max} at speeds above and below $\bar{U}_{opt}$ are included in Table 8. It can be seen that the optimized systems yield lower values for μ_{max} than the purely alternate systems, not only at all the speeds below $\bar{U}_{opt}$, but also for some speeds above $\bar{U}_{opt}$.

Effects of Higher Powers in the Objective Function

The objective function used so far involved square powers of the mistunings, as represented by Eq. (27). However, higher powers may be used in order to reduce the values of peak mistunings, especially when light mistuning is desired. The results obtained for objective functions with powers 4 and 6 (the results of powers 2 are given in Table 4), with $r=0.1$ [see Eq. (23)] indicate that, in all cases, both the optimized mistuning vectors and the optimized eigenvalues remain essentially unchanged, thus indicating that the optimum mistuning vector is insensitive to the powers in the objective function. This indicates that the design obtained by minimizing the maximum blade mistuning may be expected to be close to that obtained by minimizing the sum of the squares of individual blade mistunings.

Effects of Mistunings in the Elastic Axis Locations

The computer program was written to accommodate mistunings in elastic axis locations. The numerical results, however, showed that mistunings in the elastic axis locations yield only insignificant changes in the eigenvalues of the system. The reason for this ineffectiveness are twofold.

1) A first-order change in elastic axis location is required in order to cause a second-order change in the moments of inertia.

2) A first-order change in elastic axis location produces changes in the aerodynamic matrices which are of third order relative to the mass and stiffness terms.

Hence, any attempt to introduce mistunings through elastic axis relocations was abandoned.

Table 8 Flutter speeds and some μ_{max} values for optimized stiffness mistuning systems and the purely alternate stiffness mistuning systems, $N=12$

Description of variable	Type of mistuned system Optimized (at $\bar{U}_{opt}=0.5$) stiffness mistunings	Purely alternate stiffness mistunings ±1.621%	Optimized (at $\bar{U}_{opt}=0.787$) stiffness mistunings	Purely alternate stiffness mistunings ±4.21%
Flutter speed $\bar{U}_F$	0.83	0.93	>1.00	>1.00
μ_{max} at $\bar{U}=1.0$	—	—	−0.3501D−02	−0.3296D−02
=0.9	—	—	−0.4441D−02	−0.3395D−02
=0.7	−0.1863D−02	−0.1939D−02	−0.4235D−02	−0.2897D−02
=0.6	−0.2305D−02	−0.1972D−02	−0.2961D−02	−0.2456D−02
=0.4	−0.1571D−02	−0.1384D−02	−0.1631D−02	−0.1469D−02
=0.3	−0.1067D−02	−0.9683D−03	−0.1094D−02	−0.1055D−02

Application of Uncoupled Torsional Analysis to a General 2-D Bending-Torsion System

The analysis presented in this work considers torsional oscillations only. It may be argued that the results obtained herein are of limited value since it has been shown[3,4] that coupled bending torsion analysis has a considerable effect on the flutter speed.

At this stage one should remember that compressor blade flutter involves a system with large inertia forces and relatively small aerodynamic forces. It has been mentioned in the example presented in this work that the aerodynamic forces are two orders of magnitude smaller than the inertia forces, or stiffness forces. Hence, one cannot expect that these small aerodynamic forces can introduce any significant coupling effect between the bending and the torsion degrees of freedom. It is, therefore, reasonable to assume that any sizable bending-torsion coupling must originate from inertial effects. If this is true, then these effects can be calculated using a single blade, in the absence of aerodynamic forces, by a standard vibration analysis. For a system with no damping (or a lightly damped system) the coupled bending-torsion vibrations yield two modes of vibration with two different frequencies. Each one of these modes describes one nodal point which is fixed in space. This nodal point can be considered as being the torsional ("uncoupled") location of the elastic axis, and the anlaysis presented herein then can be applied separately to each of these two vibrational modes. The procedure just described is summarized as follows.

1) Perform a bending-torsion vibration analysis, using a single blade with no aerodynamic force.

2) Determine the nodal points of the vibrational modes computed in 1, and the coupled vibrational frequencies.

3) Perform a flutter analysis for each of the aforementioned vibrational modes, using the equivalent uncoupled torsional modes, where the nodal points determined in 2 are used as the elastic axis location. Change the radius of inertia of the system to conform with the new elastic axis location, and change the torsional stiffness to conform with the coupled frequency determined in 2.

The above procedure was tested using the rotor described earlier, with the following changes introduced.

1) The elastic axis of the blade is assumed to be at 25% chord.

2) The center of gravity is kept at midchord.

3) Bending is introduced with bending to torsion frequency ratio $\omega_h/\omega_\alpha = 1.2$.

4) Structural damping in both bending and torsion is introduced, so that $g = 0.004$.

The result for modified system is shown in Fig. 1, together with curves taken from Ref. 3. which include the above configuration, but for a compressor with 56 blades. It can be seen that the agreement between the results of Ref. 3 and the present results (based on the procedure described earlier) is excellent. Hence, the results obtained in this work can be applied to both coupled and uncoupled bending-torsion flutter, provided the equivalent uncoupled torsional system is determined following the procedure just described. The details of comparative calculation are presented in Appendix B.

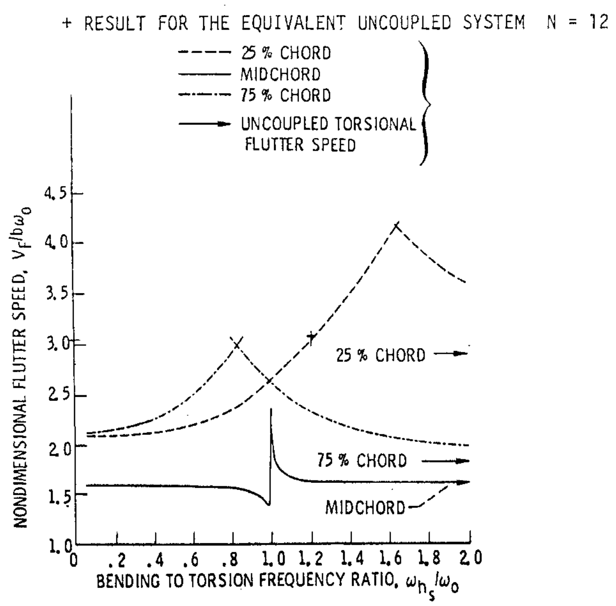

Fig. 1 Comparison between the eigenvalues of the equivalent uncoupled system and those of the coupled bending torsion system (from Refs. 3), c.g. at midchord, $g = 0.004$.

Concluding Remarks

The combined analytical-numerical work yields the most stable eigenvectors and the values of mistunings associated with these eigenvectors. It is shown that the values of mistunings which lead to the most stable eigenvectors are not unique. It is also shown that the maximal stability of the mistuned system can be predicted from the data of the tuned system only.

The numerical optimization techniques yield results which confirm those obtained analytically. It is particularly interesting to note that only 1.621% peak mistuning is required to bring μ_{max} within 10% of μ_{mean}. The multiplicities of all minima, which are shown to be equal to the number of compressor blades, helps in obtaining the global optimum, provided the starting mistuning vector is given alternate signs. The simplified, purely rotational model adopted throughout this work is shown to be valid for the more general case of bending-torsion flutter, provided an equivalent uncoupled torsional system is defined as shown herein. It is hoped that beyond the many results presented in this multipart work, some physical insight into the problem of stabilization by means of mistuning has also been gained. It is further hoped that this physical insight will be found helpful in tackling the problem of blade response to external excitation, which is currenty under study.

Appendix A: Property of N Multiplicities of all Minima

It is shown in the following that for a mistuned system with N blades, the same set of eigenvalues will be obtained by N different sets of mistunings.

Assume, for example, that the set of mistunings $a_1, a_2, \ldots, a_{N-1}, a_N$ satisfies a certain minimum condition for the mistuned system. This implies that blade 1 is mistuned by a_1, blade 2 by a_2, and so on, until blade N. However, since the choice of blade 1 is arbitrary, let us now choose the previously designated blade N as blade 1. In this case, blade 2 will represent the previously designated blade 1 and so on until the previously designated blade ($N-1$) is reached which will now be denoted as blade 1. The changes just described did not introduce any change in the physical mistunings. It follows that the same physical mistunings of the blades will now be represented by $a_N, a_1, a_2, \ldots, a_{N-2}, a_{N-1}$. This process can be repeated in a cyclic fashion to obtain mistuning sets like $a_{N-1}, a_N, a_1, a_2, \ldots, a_{N-3}, a_{N-2}$ or $a_{N-2}, a_{N-1}, a_N, a_1, a_2, \ldots, a_{N-4}, a_{N-3}$ and so on until a complete cyclic change in the mistuning set is obtained. Clearly, such a cyclic set of changes yields N sets of mistunings which must yield the same eigenvalues since they represent the same physical system. This property was tested numerically using the optimized values of mistunings, as obtained for $\bar{U} = 0.5$. In all cases the eigenvalues obtained were identical, thus confirming the aforementioned property of N multiplicities of the dynamic properties of mistuned systems.

Appendix B: Determination of Equivalent Uncoupled Torsional System, as Derived From a Coupled Bending-Torsion Configuration

The equations of motion of the blade represented by Fig. B1 with no aerodynamic force are given by

$$\left(-(\omega^2/\omega_\alpha^2)\begin{bmatrix}1 & \bar{x}_{c.g.}\\ \bar{x}_{c.g.} & \bar{r}_\alpha^2\end{bmatrix} + \begin{bmatrix}(\omega_h^2/\omega_\alpha^2) & \\ & \bar{r}_\alpha^2\end{bmatrix}\right)\begin{Bmatrix}h/c\\ \alpha\end{Bmatrix} = 0 \tag{B1}$$

where all distances are nondimensionalized with respect to the chord. Denoting

$$\lambda^2 = -(\omega^2/\omega_\alpha^2) \tag{B2}$$

it is possible to write the solution for λ^2 by

$$\lambda^2 = \left\{-\bar{r}_\alpha^2\left(1+\frac{\omega_h^2}{\omega_\alpha^2}\right) \pm \left[\bar{r}_\alpha^4\left(1+\frac{\omega_h^2}{\omega_\alpha^2}\right)^2 - 4\frac{\omega_h^2}{\omega_\alpha^2}(\bar{r}_\alpha^2 - \bar{x}_{c.g.}^2)\bar{r}_\alpha^2\right]^{1/2}\right\}\Big/2(\bar{r}_\alpha^2 - \bar{x}_{c.g.}^2) \tag{B3}$$

Consider now the case where the elastic axis is at 25% chord; $\omega_h/\omega_\alpha = 1.2$; the center of gravity is at midchord; $\bar{x}_{c.g.} = 0.25$; $\bar{r}_\alpha^2 = \bar{r}_{c.g.}^2 + (0.25)^2 = 0.14585$; $\bar{r}_\alpha = 0.3819$; and $\bar{r}_{c.g.} = 0.2887$. Substitution of these values into Eq. (B3) yields

$$\lambda_1^2 = -3.5634 \;; \qquad \lambda_1 = \pm i\ 1.8877$$

$$\lambda_2^2 = -0.70732; \qquad \lambda_2 = \pm i\ 0.841025$$

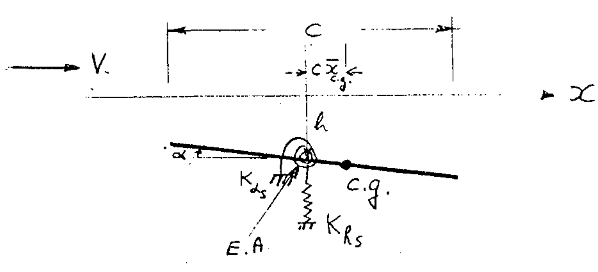

Fig. B1 Airfoil in bending torsion motion.

The mode of vibration for λ_1 is given by

$$\frac{h/c}{\bar{r}_\alpha^2(\lambda_1^2+1)} = \frac{-\alpha}{\lambda_1^2 0.25} \tag{B4}$$

Substituting the preceding values for $\bar{r}_\alpha^2$ and λ_1^2 one obtains

$$\frac{h/c}{\alpha} = -0.4195 \tag{B5}$$

Equation (B5) implies a nodal point at 0.4195c behind the elastic axis, or at 0.6695c downstream from the leading edge.

The equivalent uncoupled torsional mode will, therefore, have an elastic axis at 0.6695c from the leading edge, with $\bar{r}_\alpha$ given by

$$\bar{r}_\alpha = \sqrt{\bar{r}_{c.g.}^2 + (0.6695-0.5)^2}$$

or $\bar{r}_\alpha = 0.3348$. In addition, the structural stiffness is such as to cause the equivalent torsional stiffness to oscillate at a frequency of 1.8877 ω_α.

For comparison with Ref. 3, $\bar{U}$, as defined herein, is given by $\bar{U} = U/(1.8877\ \omega_\alpha)c$ or

$$\bar{U} = \left(\frac{U}{\omega_\alpha n}\right)\cdot\frac{1}{3.775} \tag{B6}$$

Hence, the relation between the flutter speed of the equivalent system $\bar{U}_{EF}$ and the one computed in Ref. 3 for the coupled system $\bar{U}_{CF}(=U_F/\omega_\alpha b)$ is given by Eq. (B6), that is,

$$\bar{U}_{CF} = \bar{U}_{EF} 3.775 \tag{B7}$$

Acknowledgment

The work presented herein was supported by NASA Lewis Research Center, under NASA Grant NAG-3-347.

References

[1]Nissim, E., "Optimization of Cascade Blade Mistuning, Part I: Equations of Motion and Basic Inherent Properties," *AIAA Journal*, Vol. 23, Aug. 1985, pp. 1213-1222.

[2]Miura, H. and Schmit, L. A., "NEWSUMT—A Fortran Minimization—Users Guide," NASA CR 159070, June 1979.

[3]Kaza, K. R. and Kielb, R. E., "Effects of Mistuning on Bending-Torsion Flutter and Response of a Cascade in Incompressible Flow," NASA TM-81674, April 1981.

[4]Benediksen, O. and Friedmann, P., "Coupled Bending Torsion Flutter in Cascades," *AIAA Journal*, Vol. 18, Feb. 1980.